J. Fluid Mech. (1968), *vol.* 33, *part* 1, *pp.* 87–91
Printed in Great Britain

Note on the stability of a visco-elastic liquid film flowing down an inclined plane

By A. S. GUPTA AND LAJPAT RAI
Department of Mathematics, Indian Institute of Technology, Kharagpur

(Received 28 June 1967 and in revised form 14 December 1967)

An analysis is presented for the stability of a viscoelastic liquid film flowing down an inclined plane with respect to three-dimensional disturbances. It is shown that under certain circumstances, these disturbances are more unstable than the two-dimensional ones, contrary to Squire's theorem.

1. Introduction

The stability characteristics of the flow of an ordinary viscous liquid down an inclined plane have been investigated by Benjamin (1957) and Yih (1955, 1963). The same problem has been solved by Yih (1965) for the case of a non-Newtonian fluid obeying the constitutive equation proposed by Reiner and Rivlin (see Truesdell 1962) and by Gupta (1967) for an incompressible second-order fluid whose constitutive equation is due to Coleman & Noll (1960). The model due to Reiner and Rivlin is incomplete in the sense that although the stress–strain rate relation involves the cross-viscous term, it does not take account of the acceleration gradient term which is of the same order as the cross-viscous term (Truesdell 1962). This defect is remedied in the constitutive equation of a second-order fluid proposed by Coleman & Noll, which they obtained by asymptotic expansion of a general memory functional for slow rates of deformation.

However, all these stability analyses are concerned with the response of the flow to two-dimensional disturbances. In the present note we have studied the stability characteristics of a layer of a second-order fluid flowing down an inclined plane with respect to three-dimensional disturbances. In the appendix, we point out an error in Gupta's (1967) earlier paper.

2. Mathematical formulation and stability analysis

A film of an incompressible second-order fluid of thickness d flows down a plane inclined at an angle δ to the horizon. In a rectangular system (x_1, x_2, x_3), the basic flow is parallel to x_1-axis and x_2-axis is normal to the plane directed downwards, the origin being taken on the undisturbed free surface.

We use the constitutive equation of a second-order fluid (due to Coleman & Noll 1960)

$$S_{ij} + p\delta_{ij} = \eta_0 A_{(1)ij} + \beta A_{(1)ik} A_{(1)kj} + \gamma A_{(2)ij}, \tag{2.1}$$

where η_0, β and γ are material constants and $A_{(N)ij}$ are the Rivlin–Ericksen tensors (see Gupta 1967). Substituting S_{ij} from (2.1) in the equations of momentum and continuity, Gupta deduced the velocity distribution for the undisturbed flow as $\{v_1^0(x_2), 0, 0\}$, where

$$v_1^0(x_2) = \tfrac{3}{2}v^* \left(1-\frac{x_2^2}{d^2}\right). \tag{2.2}$$

Here $v^* = pgd^2 \sin\delta/3\eta_0$ is the average velocity.

We now consider the stability of this solution with respect to three-dimensional disturbances, each component of which is a function of x_2 multiplied by $\exp[i\hat{\alpha}(x_1-\hat{c}t)+i\hat{\beta}x_3]$. Here $\hat{\alpha}$ and $\hat{\beta}$ are the wave-numbers along x_1 and x_3 axes and $\hat{c} = \hat{c}_r + ic_i$ is the phase speed of the disturbance. Let us introduce a new co-ordinate system (y_1, y_2, y_3) by rotating x_1, x_3 through an angle $\theta = \tan^{-1}\hat{\beta}/\hat{\alpha}$ keeping x_2-axis fixed. In this system

$$y_1 = x_1\cos\theta + x_3\sin\theta, \quad y_2 = x_2, \quad y_3 = x_3\cos\theta - x_1\sin\theta. \tag{2.3}$$

The stability problem under consideration is now equivalent to that of a flow with the velocity field $\{v_1^0(y_2)\cos\theta, 0, -v_1^0(y_2)\sin\theta\}$ subject to perturbations of the form

$$q_j' = q_j^*(y_2)\exp[i\alpha(y_1-ct)] \quad (j = 1, 2, 3, \ldots), \tag{2.4}$$

$q_j^*(y_2)$ being the amplitude of the perturbation.

In the co-ordinates y_1, y_2, y_3 the equations of momentum can be linearized with respect to the basic steady flow as

$$\rho\left[\frac{\partial v_1'}{\partial t} + \frac{dv_1^0}{dy_2}v_2'\cos\theta + v_1^0\frac{\partial v_1'}{\partial y_1}\cos\theta\right] = -\frac{\partial p'}{\partial y_1} + \frac{\partial D_{11}'}{\partial y_1} + \frac{\partial D_{12}'}{\partial y_2}, \tag{2.5}$$

$$\rho\left[\frac{\partial v_2'}{\partial t} + \frac{\partial v_2'}{\partial y_1}v_1^0\cos\theta\right] = -\frac{\partial p'}{\partial y_2} + \frac{\partial D_{12}'}{\partial y_1} + \frac{\partial D_{22}'}{\partial y_2}, \tag{2.6}$$

$$\rho\left[\frac{\partial v_3'}{\partial t} + \frac{\partial v_3'}{\partial y_1}v_1^0\cos\theta - v_2'\frac{dv_1^0}{dy_2}\sin\theta\right] = \frac{\partial D_{13}'}{\partial y_1} + \frac{\partial D_{23}'}{\partial y_2}, \tag{2.7}$$

where dashes refer to perturbation quantities, p' is the perturbation pressure and $D_{ij} = S_{ij} + p\delta_{ij}$ is the deviatoric part of the stress tensor S_{ij}.

The equation of continuity reduces to

$$\frac{\partial v_1'}{\partial y_1} + \frac{\partial v_2'}{\partial y_2} = 0. \tag{2.8}$$

We shall choose v^* as unit of velocity, d as unit of length, d/v^* as unit of time and ρv^{*2} as unit of stresses and retain the same notations for non-dimensional variables. Equation (2.8) then permits the use of a non-dimensional stream function ψ' defined by $v_1' = \partial\psi'/\partial y_2$, $v_2' = -\partial\psi'/\partial y_1$. Elimination of pressure between (2.5) and (2.6) leads to an equation which may involve v_3' in its right-hand side, and Squire's theorem ceases to hold. Assuming

$$\psi' = \phi(y_2)\exp[i\alpha(y_1-ct)], \quad v_3' = \xi(y_2)\exp[i\alpha(y_1-ct)], \tag{2.9}$$

and using (2.1) and (2.8) we can write the final equations for this stability problem in terms of non-dimensional variables as follows

$$\begin{aligned} &Ri\alpha[(v_1^0\cos\theta - c)(D^2-\alpha^2)\phi - \phi D^2 v_1^0\cos\theta] \\ &\quad = (D^2-\alpha^2)^2\phi + i\alpha RM[(v_1^0\cos\theta - c)(D^2-\alpha^2)^2\phi + 2\,Dv_1^0(D^2-\alpha^2)\xi\sin\theta] \\ &\qquad + \tfrac{1}{2}Ri\alpha N[Dv_1^0(D^2-\alpha^2)\xi\sin\theta], \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&Ri\alpha[(v_1^0\cos\theta-c)\xi+Dv_1^0\phi\sin\theta]\\
&\quad=(D^2-\alpha^2)\xi+Ri\alpha M[(v_1^0\cos\theta-c)(D^2-\alpha^2)\xi+2Dv_1^0D\xi\cos\theta+\xi D^2v_1^0\cos\theta\\
&\qquad-\alpha^2Dv_1^0\phi\sin\theta+i\alpha D^2v_1^0\phi\sin\theta+i\alpha Dv_1^0D\phi\sin\theta\\
&\qquad+D^2v_1^0D\phi\sin\theta]+i\alpha RN[\tfrac{1}{2}Dv_1^0(D^2-\alpha^2)\phi\sin\theta\\
&\qquad+D^2v_1^0D\phi\sin\theta+\tfrac{1}{2}D^2v_1^0\xi\cos\theta+Dv_1^0D\xi\cos\theta]. \qquad (2.11)
\end{aligned}
$$

Here (2.10) is obtained by eliminating pressure between (2.5) and (2.6) and (2.11) follows from (2.7) and $D\equiv d/dy_2$. Further, R, M and N are respectively the Reynolds number, visco-elastic and cross-viscous parameters defined as

$$
R=\frac{v^*d\rho}{\eta_0},\quad M=\frac{\gamma}{d^2\rho},\quad N=\frac{2\beta}{d^2\rho}. \qquad (2.12)
$$

The no-slip boundary condition at the plane gives

$$
\phi(1)=D\phi(1)=\xi(1)=0. \qquad (2.13)
$$

As for the boundary condition at the free surface, we note that the tangential stress must vanish there ($S_{12}=S_{23}=0$) and the normal stress must balance that due to surface tension. Proceeding as in Gupta's (1967) paper, we give below the final forms of these boundary conditions as

$$
(D^2+\alpha^2)\phi+i\alpha RM[3\phi\cos\theta-c^*(D^2+\alpha^2)\phi]-\frac{3}{c^*}\phi\cos\theta=0 \quad\text{at}\quad y_2=0, \qquad (2.14)
$$

$$
D\xi=-\frac{3}{c^*}\sin\theta\phi \quad\text{at}\quad y_2=0, \qquad (2.15)
$$

and

$$
\begin{aligned}
&(D^2-\alpha^2)D\phi+i\alpha RM[6\xi\sin\theta-3D\phi\cos\theta-c^*(D^2-\alpha^2)D\phi]\\
&\qquad+\tfrac{3}{2}i\alpha RN\xi\sin\theta+[i\alpha Rc^*-2\alpha^2+2i\alpha^3c^*MR]D\phi\\
&\qquad+\frac{i\alpha}{c^*}(RS\alpha^2+3\cot\delta)\phi=0 \quad\text{at}\quad y_2=0, \qquad (2.16)
\end{aligned}
$$

where $c^*=c-\frac{3}{2}\cos\theta$ and S is the non-dimensional parameter characterizing surface tension.

Equations (2.10) and (2.11), along with the boundary conditions (2.13) to (2.16), constitute an eigenvalue problem. For a non-trivial solution, a relation

$$
c=c(R,M,N,S,\alpha,\theta) \qquad (2.17)
$$

will hold good among the physical parameters and this will determine the neutral stability curves $c_i=0$.

3. Solution for long waves

Since the equations and the boundary conditions do not degenerate as $\alpha\to 0$, we may solve this system following the regular perturbation procedure of Yih (1963).

For the zeroth approximation (i.e. for terms of order α^0), we have

$$
D^4\phi_0=D^2\xi_0=0,
$$

subject to the boundary conditions

$$
\left.
\begin{aligned}
&\phi_0(1)=D\phi_0(1)=\xi_0(1)=0,\quad D^2\phi_0(0)=\frac{3\cos\theta}{c^*},\\
&D\xi(0)=-\frac{3\phi_0(0)\sin\theta}{c^*},\quad D^3\phi_0(0)=0.
\end{aligned}
\right\} \qquad (3.1)
$$

The solution of this differential system is

$$\left.\begin{aligned}\phi_0(y_2) &= (1-y_2)^2, \quad c_0 = 3\cos\theta, \\ \xi_0(y_2) &= 2(1-y_2)\tan\theta,\end{aligned}\right\} \tag{3.2}$$

where c_0 is the eigenvalue for the zeroth approximation. Since c_0 is real we may conclude that $\alpha = 0$ is a part of the neutral stability curve.

For the next approximation, we retain terms of order α and we get upon using (3.2),

$$D^4\phi_1 = -6i\alpha R y_2 \cos\theta,$$
$$D^2\xi_1 = 3i\alpha R[-1+y_2^2-4My_2+N(1+2y_2)],$$

subject to

$$\left.\begin{aligned}&\phi_1(1) = D\phi_1(1) = \xi_1(1) = 0,\\ &D^2\phi_1(0) - 2\phi_1(0)+\tfrac{4}{3}\Delta c\sec\theta = 0,\\ &D\xi_1 = -3\sin\theta(\phi_1/c_0^* - \phi_0\Delta c/c_0^{*2}) \quad \text{at} \quad y_2 = 0,\\ &D^3\phi_1 + 6i\alpha RM(\sec\theta+\sin\theta\tan\theta) + 3i\alpha RN\sec\theta\tan\theta\\ &\qquad - 3i\alpha R\cos\theta + 2i\alpha\sec\theta\cot\delta = 0 \quad \text{at} \quad y_2 = 0,\end{aligned}\right\} \tag{3.3}$$

where $c_0^* = c_0 - \frac{3}{2}\cos\theta$ and Δc stands for the change in c_0 as α deviates from zero.

It may be noted that the differential system for ϕ_1 does not involve ξ_1 and consequently this suffices to determine Δc as follows

$$\tfrac{2}{3}\Delta c\sec\theta = \tfrac{4}{5}i\alpha R\cos\theta - \tfrac{2}{3}i\alpha R\sec\theta\cot\delta - 2i\alpha RM[\sec\theta+\sin\theta\tan\theta] - i\alpha RN\sin\theta\tan\theta. \tag{3.4}$$

Neutral stability curves ($c_i = 0$) in (α, R)-plane are given by $\Delta c = 0$, which determines the critical Reynolds number as

$$R_c = \tfrac{2}{3}\cot\delta/[\tfrac{4}{5}\cos^2\theta - 2M(1+\sin^2\theta) - N\sin^2\theta]. \tag{3.5}$$

This shows that for three-dimensional disturbances ($\theta \neq 0$), R_c is affected both by M and N. For two-dimensional disturbances ($\theta = 0$), (3.5) gives the critical Reynolds number R_{ct} (say) as

$$R_{ct} = \tfrac{2}{3}\cot\delta/(\tfrac{4}{5} - 2M), \tag{3.6}$$

which does not involve the cross-viscous parameter N. Now $R_{ct} > R_c$ if

$$\tfrac{4}{5}\cos^2\theta - 2M(1+\sin^2\theta) - N\sin^2\theta > \tfrac{4}{5} - 2M,$$

which implies $$-2M > \tfrac{4}{5} + N \quad \text{provided} \quad \theta \neq 0. \tag{3.7}$$

When $M = 0$, the above inequality is not valid if $N > 0$. Thus for a Reiner–Rivlin fluid with a positive cross-viscous coefficient two-dimensional disturbances will be more unstable than three-dimensional ones, in agreement with the conclusion of Listrov (1966). However, for a visco-elastic fluid with $M < 0$ the inequality (3.7) can clearly hold good with $N > 0$ if $|M| > \frac{2}{5} + \frac{1}{2}N$, in which case oblique disturbances will be more unstable than the two-dimensional ones. Under such circumstances, therefore, Squire's theorem no longer remains valid.

Appendix

In the paper by Gupta (1967), the boundary condition (3.24) on the normal stress is in error. The corrected equation is

$$(1-RMi\alpha c')\,\phi'''(0)+[3iM\alpha^3c'R-i\alpha MR+i\alpha c'R-3\alpha^2]\,\phi'(0)+\left[\frac{i\alpha}{c'}\cot\beta_0+\frac{i\alpha^3}{c'}SR\right]\phi(0)=0.$$

The differential equation (3.18) and the remaining boundary conditions are all correct. With this modification, the critical Reynolds number R_c (given by (4.10) of Gupta's (1967) paper) now stands corrected as

$$R_c=5\cot\beta_0/(2-5M).$$

However, the qualitative conclusions about instability remain unaffected.

REFERENCES

BENJAMIN, T. B. 1957 *J. Fluid Mech.* **2**, 554.
COLEMAN, B. D. & NOLL, W. 1960 *Arch. Rat. Mech. Anal.* **6**, 355.
GUPTA, A. S. 1967 *J. Fluid Mech.* **28**, 17.
LISTROV, A. T. 1966 *Soviet Phys. Dokl.* (English translation), **10**, 921.
TRUESDELL, C. 1962 *Proc. Int. Symp., Haifa, Israel*, 23–27 *April*, p. 1.
YIH, C. S. 1955 *Proc. 2nd U.S. Nat. Congr. Appl. Mech.* pp. 623–8.
YIH, C. S. 1963 *Phys. Fluids*, **6**, 321.
YIH, C. S. 1965 *Phys. Fluids*, **8**, 1257.